**Indian Statistical Institute, Bangalore**
**B. Math (II) Second Semester 2018-19**
**Backpaper Examination : Statistics (II)**

**Date: 13-06-2019** **Maximum Score 100** **Duration: 3 Hours**

1. Let $X_1,\ X_2,\cdots,X_n$ be a random sample from $Poisson(\lambda)$ , $\lambda > 0$. Find *method of moments estimator* as well as *maximum likelihood estimator* for $\lambda$. What happens if $\sum_{i=1}^{n} X_i = 0$?

   $[5 + 7 + 2 = 14]$

2. Suppose there are 12 reservation counters for state run buses in the city of Bangalore. A counter is open on any particular day with probability $\theta$, $0 < \theta < 1$. The counters function independently of each other. For reasons of proximity and convenience, Shilpak uses either counter 1 or counter 2. Shilpak is interested in knowing *i)* $\tau_1(\theta)$, the probability that either counter 1 is open or counter 2 is open on any given day and *ii)* $\tau_2(\theta)$, the probability that exactly one of the counters 1 and 2 is open on a given day. Let $X_i = 1$ if the *ith* counter is open and $X_i = 0$ if the *ith* counter is closed on a given day, $1 \le i \le 12$. Let $X_1,\ X_2,\cdots,X_{12}$ be the random sample taken on some day indicating whether various of the reservation counters are open or not.

   (a) Find $\tau_1(\theta)$ and $\tau_2(\theta)$.
   (b) Show that $T = \sum_{i=1}^{12} X_i$ is a minimal sufficient statistic for $\theta$.
   (c) Is $T = \sum_{i=1}^{12} X_i$ complete as well? Substantiate.
   (d) Find Fisher information $I(\theta)$ contained in the sample $X_1,\ X_2,\cdots,X_{12}$ about $\theta$.
   (e) Find an unbiased estimator for $\tau_2(\theta)$. Hence or otherwise obtain *UMVUE* for $\tau_2(\theta)$.

   $[4 + 4 + 5 + 5 + 8 = 26]$

3. Suppose that an electronic system contains $n$ similar components which function independently of each other and which are connected in series, so that the system fails as soon as one of the components fails. Suppose $X_1,\ X_2,\cdots,X_n$ denote the lifetimes of the $n$ components. Suppose also that the lifetime of each component, measured in hours, has *exponential distribution* with *pdf* $\frac{1}{\lambda}e^{-\frac{x}{\lambda}}I_{(0,\infty)}(x)$, $\lambda > 0$. The system user has reasons to believe that $\lambda$ has a *prior distribution* given by $Gamma(a, b)$, $a > 0$, $b > 0$ known.

   (a) Find the distribution of $Y$, the lifetime of the system. Determine $E(Y)$ $(= \theta$ say), the expected lifetime of the system.
   (b) Obtain *posterior distribution* of $\theta$ given the observation $Y$. Obtain mean and variance of the *posterior distribution* of $\theta$.
   (c) Suggest *Bayes estimator* for $\theta$.

   $[(6 + 3) + 10 + 3 = 22]$

[PTO]

4. Let $X_1,\ X_2, \cdots, X_n$ be the random sample from $N(\theta, \sigma^2)$, $\theta \in I\!R$ and $\sigma^2 > 0$ are both unknown. Consider the testing problem

$$H_0 : \theta = \theta_0 \ versus \ H_1 : \theta \neq \theta_0.$$

where $\theta_0 \in I\!R$ is a specified value.

(a) Derive *size* $\alpha$ *likelihood ratio test.*

(b) Find $p$-value.

(c) Find 90% *confidence interval for* $\theta$.

[$11 + 3 + 6 = 20$]

5. An automatic filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $\boldsymbol{s}^2 = 0.0153$ square fluid ounces. If $\sigma^2$, the variance of fill volume, exceeds 0.01 square fluid ounces, an unacceptable proportion of bottles will be under or overfilled.

(a) Is there evidence in the sample data to suggest that the manufacturer has a problem with under and overfilled bottles? Use $\alpha = 0.05$.

(b) Report the $p - value$.

(c) Obtain 90% one sided confidence interval for $\sigma^2$.

[$12 + 4 + 6 = 22$]